

Applications of He's Amplitude-Frequency Formulation to the Free Vibration of Strongly Nonlinear Oscillators

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Abstract

In this paper, He's amplitude-frequency formulation is used to obtain a periodic solution of a nonlinear oscillator. We illustrate that He's amplitude-frequency formulation is very effective and convenient and does not require linearization or small perturbation. The obtained results are valid for the whole solution domain with high accuracy.

Keywords: Periodic solution, He's amplitude-frequency formulation, Nonlinear oscillators

1- Introduction

Very recently, various kinds of analytical methods and numerical methods have been used to handle the nonlinear problems without possible small parameters.

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Traditional perturbation methods have many shortcomings and they are not valid for strongly nonlinear oscillators. To overcome the shortcomings, many new techniques have been proposed for dealing with the nonlinear oscillators, for example, the variational iteration method [1-3], the homotopy perturbation method [4-7], and the energy balance method [8-11]. The aim of the work is to show how to apply the amplitude-frequency formulation developed recently by He [12-13] and used later by many authors [14-19] to solve the nonlinear oscillator's problems.

2- He's amplitude-frequency formulation

We consider a generalized nonlinear oscillator in the form:

$$\ddot{u} + f(u) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \quad (1)$$

It can be seen that since no small parameter exists in equation (1), besides the equation involves discontinuity, we cannot apply the traditional perturbation methods directly. Because of the fact that the amplitude-frequency formulation does not require a small parameter and a linear term in the differential equation as well, Eq. (1) can be approximately solved using amplitude-frequency formulation. According to amplitude-frequency formulation for Eq. (1), we use two trial functions $u_1 = A \cos \omega_1 t$, and $u_2 = A \cos \omega_2 t$, where ω_1 and ω_2 can be freely chosen, generally we choose $\omega_1 = 1$ where $\omega_2 = \omega$, and ω is the frequency of the nonlinear oscillator. Substituting u_1 and u_2 into Eq. (1) we obtain, respectively, the following residuals.

$$R_1(t) = -\omega_1^2 A \cos \omega_1 t + f(A \cos \omega_1 t), \quad (2)$$

and

$$R_2(t) = -\omega_2^2 A \cos \omega_2 t + f(A \cos \omega_2 t). \quad (3)$$

Ji-Huan He suggested the following amplitude-frequency formulation in Ref. [20]

$$\omega^2 = \frac{\omega_1^2 R_2(0) - \omega_2^2 R_1(0)}{R_2(0) - R_1(0)}. \quad (4)$$

Geng and Cai [21] suggested a modification, which is

$$\omega^2 = \frac{\omega_1^2 R_2\left(\frac{T_2}{N}\right) - \omega_2^2 R_1\left(\frac{T_1}{N}\right)}{R_2\left(\frac{T_2}{N}\right) - R_1\left(\frac{T_1}{N}\right)}. \quad (5)$$

where T_1 and T_2 are the periods of the trial solutions, u_1 and u_2 , respectively, N is generally chosen as $N = 12$. and the phase of the residuals is $\pi/6$.

In 2008, Ji-Huan He improved the formulation, which reads [12, 13]

$$\omega^2 = \frac{\omega_1^2 \tilde{R}_2 - \omega_2^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1}. \quad (6)$$

Where

$$\tilde{R}_1(t) = \frac{4}{T_1} \int_0^{T_1/4} R_1(t) \cos t dt, \tag{7}$$

and

$$\tilde{R}_2(t) = \frac{4}{T_2} \int_0^{T_2/4} R_2(t) \cos \omega t dt. \tag{8}$$

3- Applications

In order to assess the accuracy of He's amplitude-frequency formulation for solving nonlinear equations and to compare it with the numerical solution, we will consider the following examples.

3-1 Example 1

Tapered beams can model engineering structures which require a variable stiffness along the length, such as moving arms and turbine blades [22-24]. In dimensionless form, the governing differential equation corresponding to fundamental vibration mode of a taper beam is given by [24]

$$\left(\frac{d^2 u}{dt^2} \right) + u + \varepsilon_1 \left(u^2 \left(\frac{d^2 u}{dt^2} \right) + u \left(\frac{du}{dt} \right)^2 \right) + \varepsilon_2 u^3 = 0, \quad u(0) = A, \quad \frac{du}{dt} = 0, \tag{9}$$

where u is displacement and ε_1 and ε_2 are arbitrary constants.

According to He's amplitude-frequency formulation, we choose two trial functions $u_1(t) = A \cos t$ and $u_2(t) = A \cos \omega t$, where ω is assumed to be the frequency of the nonlinear oscillator. Substituting the above trial functions into Eq. (9) results in, respectively, the following residuals

$$R_1(t) = \frac{A^3}{4} (-2\varepsilon_1 + 3\varepsilon_2) \cos t - \frac{A^3}{4} (2\varepsilon_1 - \varepsilon_2) \cos 3t, \tag{10}$$

and

$$R_2(t) = \frac{A}{4} (4 + 3\varepsilon_2 A^2 - 4\omega^2 - 2\varepsilon_1 \omega^2 A^2) \cos \omega t + \frac{A^3}{4} (\varepsilon_2 - 2\varepsilon_1 \omega^2) \cos 3\omega t. \tag{11}$$

If, by chance, u_1 or u_2 , is chosen to be the exact solution, then the residual, Eq. (10) or Eq. (11), is vanishing completely. In order to use He's amplitude-frequency formulation, we set

$$\tilde{R}_1(t) = \frac{4}{T_1} \int_0^{T_1/4} R_1(t) \cos t dt = \frac{A^3}{8} [-2\varepsilon_1 + 3\varepsilon_2], \tag{12}$$

and

$$\tilde{R}_2(t) = \frac{4}{T_2} \int_0^{T_2/4} R_2(t) \cos \omega t dt = \frac{A}{8} [4 - 4\omega^2 + 3\varepsilon_2 A^2 - 2\varepsilon_1 \omega^2 A^2], \quad (13)$$

where $T_1 = 2\pi$ and $T_2 = 2\pi / \omega$.

Applying He's amplitude-frequency formulation, we have

$$\omega^2 = \frac{\omega_1^2 \tilde{R}_2 - \omega_2^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1} = \frac{4 + 3\varepsilon_2 A^2}{4 + 2\varepsilon_1 A^2}, \quad (14)$$

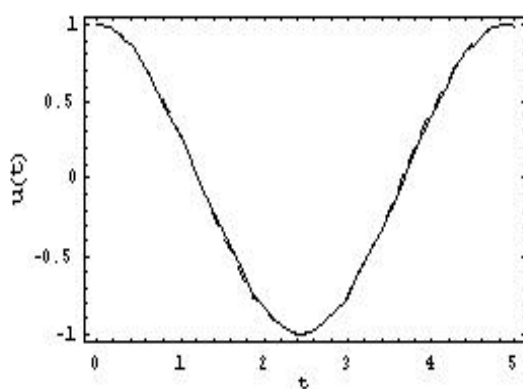
i. e.

$$\omega_{AFF} = \sqrt{\frac{4 + 3\varepsilon_2 A^2}{4 + 2\varepsilon_1 A^2}}. \quad (15)$$

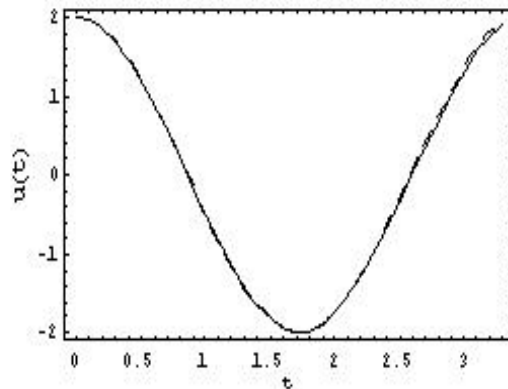
Hence, the approximate solution can be readily obtained

$$u(t) = A \cos \left(\sqrt{\frac{4 + 3\varepsilon_2 A^2}{4 + 2\varepsilon_1 A^2}} t \right). \quad (16)$$

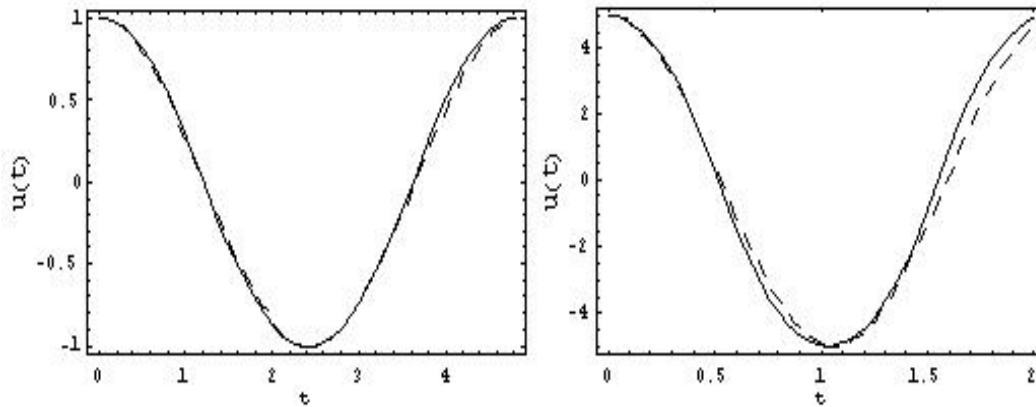
which agrees very well with the numerical integration solution using the fourth-order Runge-Kutta method as shown in Fig. 1 (a-d).



(a) $\varepsilon_1 = 0.1$; $\varepsilon_2 = 1$; $A = 1$



(b) $\varepsilon_1 = 0.1$; $\varepsilon_2 = 1$; $A = 2$



(c) $\epsilon_1 = 1; \epsilon_2 = 2; A = 1$

(d) $\epsilon_1 = 0.1; \epsilon_2 = 1; A = 5$

Fig. 1 Comparison of the approximate solution (- -) with the numerical solution (—).

3-2 Example 2

It is known that the free vibrations of an autonomous conservative oscillator with inertia and static type fifth-order non-linearities is expressed by [25-27]:

$$\frac{d^2u}{dt^2} + \lambda u + \epsilon_1 u^2 \frac{d^2u}{dt^2} + \epsilon_1 u \left(\frac{du}{dt}\right)^2 + \epsilon_2 u^4 \frac{d^2u}{dt^2} + 2\epsilon_2 u^3 \left(\frac{du}{dt}\right)^2 + \epsilon_3 u^3 + \epsilon_4 u^5 = 0. \quad (17)$$

The initial conditions for Eq. (17) are given by $u(0) = A$ and $du/dt = 0$, where A represents amplitude of the oscillation. Motion is assumed to start from the position of maximum displacement with zero initial velocity. λ is an integer which may take values of $\lambda = 1, 0$ or -1 , and $\epsilon_1, \epsilon_2, \epsilon_3$, and ϵ_4 are positive parameters.

Similarly we choose $u_1(t) = A \cos t$ and $u_2(t) = A \cos \omega t$ as trial functions, this leads to the following residuals

$$R_1(t) = \frac{A}{16} \left[(-16 - 8A^2\epsilon_1 - 6A^4\epsilon_2 + 12A^2\epsilon_3 + 10A^4\epsilon_4 + 16\lambda) \cos t - (8A^2\epsilon_1 + 7A^4\epsilon_2 - 4A^2\epsilon_3 - 5A^4\epsilon_4) \cos 3t - (3A^4\epsilon_2 - A^4\epsilon_4) \cos 5t \right] \quad (18)$$

and

$$R_2(t) = \frac{A}{16} \left[(12A^2\epsilon_3 + 10A^4\epsilon_4 + 16\lambda - 16\omega^2 - 8A^2\epsilon_1\omega^2 - 6A^4\epsilon_2\omega^2) \cos \omega t + (4A^2\epsilon_3 + 5A^4\epsilon_4 - 8A^2\epsilon_1\omega^2 - 7A^4\epsilon_2\omega^2) \cos 3\omega t + (A^4\epsilon_4 - 3A^4\epsilon_2\omega^2) \cos 5\omega t \right] \quad (19)$$

In view of Eq. (5), we have

$$\omega^2 = \frac{\omega_1^2 R_2(\frac{T_2}{N}) - \omega_2^2 R_1(\frac{T_1}{N})}{R_2(\frac{T_2}{N}) - R_1(\frac{T_1}{N})} = \frac{12A^2 \varepsilon_3 + 9A^4 \varepsilon_4 + 16\lambda}{16 + 8A^2 \varepsilon_1 + 3A^4 \varepsilon_2}, \tag{20}$$

i. e.

$$\omega_{AFF} = \sqrt{\frac{12A^2 \varepsilon_3 + 9A^4 \varepsilon_4 + 16\lambda}{16 + 8A^2 \varepsilon_1 + 3A^4 \varepsilon_2}}. \tag{21}$$

Hence, the approximate solution can be readily obtained

$$u(t) = A \cos\left(\sqrt{\frac{12A^2 \varepsilon_3 + 9A^4 \varepsilon_4 + 16\lambda}{16 + 8A^2 \varepsilon_1 + 3A^4 \varepsilon_2}} t\right). \tag{22}$$

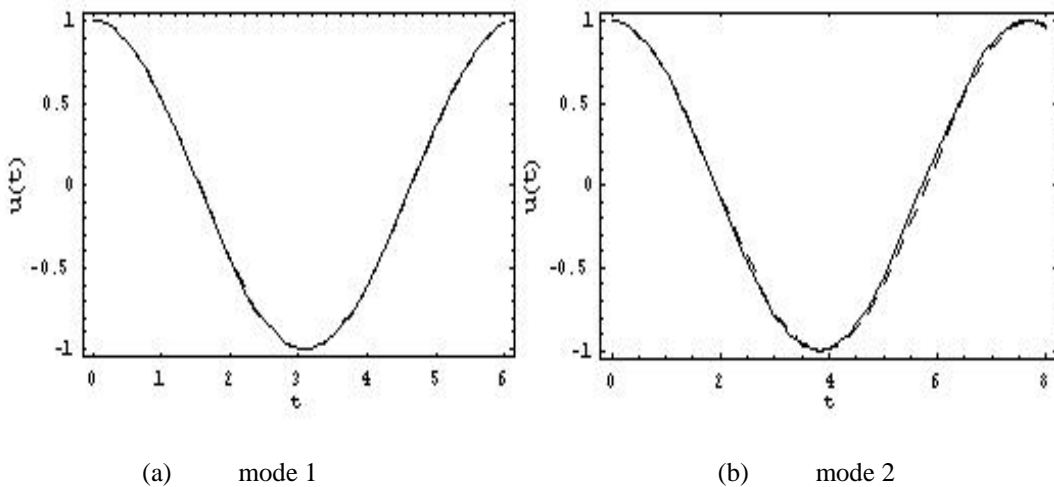
Table 1

Comparison of He's amplitude-frequency formulation (AFF) with energy balance method (EBM) [27] for $\lambda = 1, A = 1$

Mode	ε_1	ε_2	ε_3	ε_4	ω_{AFF}	ω_{EBM} [27]
1	0.326845	0.129579	0.232598	0.087584	1.01504	1.01235
2	1.642033	0.913055	0.313561	0.204297	0.823214	0.81295
3	4.051486	1.665232	0.281418	0.149677	0.622926	0.614174
4	8.205578	3.145368	0.272313	0.133708	0.474087	0.466614

The values of dimensionless parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 associated with each of the four calculation modes are shown in Table 1 [25].

Additionally, the comparison between these methodologies can be found in Fig. 2(a-d). It has been shown that the results of amplitude-frequency formulation are in good agreement with those obtained from the results of energy balance method [27] as shown in table 1 and Fig. 2(a-d).



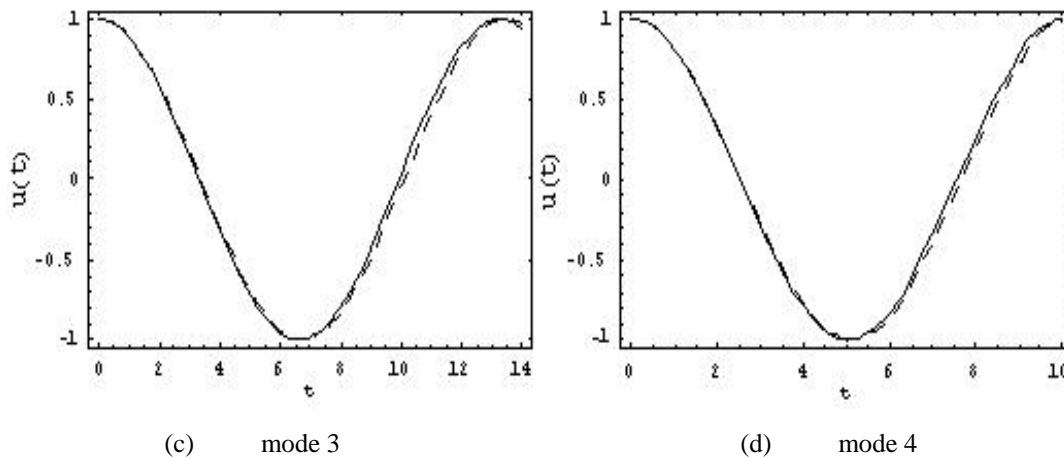


Fig. 2 The Comparison between The results of AFF (- - -) and EBM (—)

with $\lambda=1$, $A=1$ for modes 1-4.

4- Conclusion

In this work, He's amplitude-frequency formulation is proved to be a very ingenious and effective method for solving nonlinear oscillator problems. We showed that the analytical approximation is obtained easily and elegantly by this method. The analytical approximation obtained by this new method is valid for the whole solution domain with high accuracy.

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